DE GRUYTER
OPEN

DOI 10.1515/tmj-2017-0005

Hardy-type inequalities for generalized fractional integral operators

Sajid Iqbal[1], Josip Pečarić[2], Muhammad Samraiz[3] and Živorad Tomovski[4]

[1]SIqbal-Department of Mathematics, University of Sargodha (Sub-Campus Bhakkar), Bhakkar, Pakistan
[2]SJ. Pečarić-Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
[3]SM. Samraiz-Department of Mathematics, University of Sargodha, Sargodha, Pakistan
[4]SFaculty of Mathematics and Natural Sciences, Gazi Baba bb, 1000 Skopje, Macedonia; Zivorad Tomovski Department of Mathematics, University of Rijeka Radmile Matejcic 2, 51000 Rijeka, Croatia

E-mail: sajid_uos2000@yahoo.com[1], pecaric@element.hr[2], msamraiz@uos.edu.pk[3], tomovski@pmf.ukim.mk[4], zivorad.tomovski@math.uniri.hr[4]

Abstract

The aim of this research paper is to establish the Hardy-type inequalities for Hilfer fractional derivative and generalized fractional integral involving Mittag-Leffler function in its kernel using convex and increasing functions.

2010 Mathematics Subject Classification. **26D15**. 26D10, 26A33
Keywords. Hilfer fractional derivative; Mittag-Leffler function; Fractional integral.

1 Introduction

The Hardy inequality states that:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,\ p>1,$$

where equality holds only if $f \equiv 0$ introduced by G. H. Hardy in [1]. It is one of the most important inequality of analysis. Such an inequality is broadly applied to various fascinating problems in partial differential equations such as eigenvalue and boundary value problems. It also been studied for vector field as well. A variety of mathematicians [2–7] awarded the generalizations and improvements of Hardy's inequality. In this paper, we establish some more general inequalities of G. H. Hardy given in [6, 7] and applications of such inequalities for Hilfer fractional derivative and generalized fractional integral containing Mittag-Leffler function in its kernel via convex and increasing functions. We first need the following basic definition of convex function and elementary information about a particular class of function.

The following definition is given in [8].

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\varphi : I \to \mathbb{R}$ is called convex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y), \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. The function φ is strictly convex if inequality (1.1) holds strictly for all distinct points in I and $\lambda \in (0,1)$.

Received by the editors: 11 May 2015.
Accepted for publication: 14 June 2016.

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) := \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

and A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), \qquad x \in \Omega_1. \tag{1.2}$$

The upcoming result is given in [6].

Theorem 1.2. Let u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K is defined on Ω_1 by (1.2). Assume that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty.$$

If $\varphi : (0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x) \varphi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \le \int_{\Omega_2} v(y) \varphi(|f(y)|) d\mu_2(y) \tag{1.3}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$.

By substituting $k(x,y)$ by $k(x,y)f_2(y)$ and f by $\dfrac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1,2)$ are measurable functions in Theorem 1.2, then the following result is obtained (see [9, p. 220]).

Theorem 1.3. Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(f_i)$, $(i = 1,2)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1 and k a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{f_2(y)k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by

$$v(y) := f_2(y) \int_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)} d\mu_1(x) < \infty.$$

If $\varphi : (0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x) \varphi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \le \int_{\Omega_2} v(y) \varphi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) d\mu_2(y)$$

holds.

The upcoming result is given in [7].

Theorem 1.4. Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$. Let $0 < p \leq q < \infty$, K be defined on Ω_1 and the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, then v is defined as:

$$v(y) := \left[\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} d\mu_1(x) \right]^{\frac{p}{q}} < \infty.$$

If φ is non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left[\int_{\Omega_1} u(x) \left(\varphi \left(A_k f(x) \right) \right)^{\frac{q}{p}} d\mu_1(x) \right]^{\frac{1}{q}} \leq \left[\int_{\Omega_2} v(y) \varphi \left(f(y) \right) d\mu_2(y) \right]^{\frac{1}{p}}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $Imf \subseteq I$.

Next result is represented in [7].

Theorem 1.5. Let $g_i \in U(f_i), (i = 1, 2, 3)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, then v is defined by

$$v(y) := f_2(y) \int_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)} dx < \infty.$$

If $\varphi : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x) \varphi \left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \varphi \left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right) d\mu_2(y)$$

holds.

Next we give the well known definition of Riemann-Liouville fractional integrals (see [10, p. 69-71]).

Definition 1.6. Let $[a, b]$ be a finite interval on $\mathbb{R}$. The left and right sided Riemann-Liouville fractional integrals $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b-} f$ of order $\alpha > 0$ are defined as:

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) dy, x > a,$$

and

$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha - 1} f(y) dy, x < b,$$

respectively. Here Γ represents Gamma function.

The paper is organized as follows: After introduction in Section 2, we present the Hardy-type inequalities for Hilfer fractional derivative. Section 3 consists of results for generalized fractional integral which involve the Mittag-Leffler function in its kernel.

2 Hardy-type Inequalities for Hilfer fractional derivative

In this section, we give the basic definitions of Hilfer fractional derivative, then we present Hardy-type inequalities for the said derivative. But before this, we first recall the well known definition of absolutely continuous function (see [7, p. 9]).

Definition 2.1. Let $0 < a < b \leq \infty$. By $C^n[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$, we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in AC[a,b]$.

Let us now recall the definition of Hilfer fractional derivative given in [11].

Definition 2.2. Let $f \in L^1[a,b], f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 < \nu \leq 1$ with respect to $x \in [a,b]$ is defined by

$$\left(D^{\mu,\nu}_{a+} f\right)(x) := I^{\nu(1-\mu)}_{a+} \frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+} f(x)\right), \tag{2.1}$$

whenever the right hand side exists. The derivative (2.1) is usually called Hilfer fractional derivative.

The more general integral representation of equation (2.1) given in [12] define as: Let $f \in L^1[a,b], f * K_{(1-\nu)(n-\mu)} \in AC^n[a,b], n-1 < \mu < n, 0 < \nu \leq 1, n \in \mathbb{N}$, then the following equation holds true:

$$\left(D^{\mu,\nu}_{a+} f\right)(x) = \left(I^{\nu(n-\mu)}_{a+} \frac{d^n}{dx^n}\left(I^{(1-\nu)(n-\mu)}_{a+} f(x)\right)\right). \tag{2.2}$$

Specially for $\nu = 0$, $D^{\mu,0}_{a+} f = D^{\mu}_{a+} f$ is a Riemann- Liouvile fractional derivative of order μ and for $\nu = 1$ it is a Caputo fractional derivative $D^{\mu,1}_{a+} f = ^C D^{\mu}_{a+} f$ of order μ. Applying the properties of Riemann-Liouvile integral the relation (2.2) can be rewritten in the form:

$$\begin{aligned}\left(D^{\mu,\nu}_{a+} f\right)(x) &= \left(I^{\nu(n-\mu)}_{a+}\left(\left(D^{n-(1-\nu)(n-\mu)}_{a+} f\right)(x)\right)\right)\\ &= \frac{1}{\Gamma\left(\nu\left(n-\mu\right)\right)}\int_a^x (x-y)^{\nu(n-\mu)-1}\left(\left(D^{\mu+\nu(n-\mu)}_{a+} f\right)(y)\right)dy.\end{aligned}$$

Our first result is given in upcoming theorem.

Theorem 2.3. Let $n-1 < \mu < n$, $0 < \nu \leq 1$, $n \in \mathbb{N}$, $0 < p < \frac{1}{1-\nu(n-\mu)}$, $q > 1$. If $D^{\mu+\nu(n-\mu)}_{a+} f \in L^q(a,b)$, then the following inequality holds true:

$$\int_a^b \left|\left(D^{\mu,\nu}_{a+} f\right)(x)\right|^q dx \leq C \int_a^b \left|\left(D^{\mu+\nu(n-\mu)}_{a+} f\right)(y)\right|^q dy, \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $C = \frac{1}{(\Gamma(\nu(n-\mu)))^q} \frac{(b-a)^{q\nu(n-\mu)}}{((\nu(n-\mu)-1)p+1)^{q/p}(q\nu(n-\mu))}$.

Proof. Since

$$\left|\left(D_{a+}^{\mu,\nu}f\right)(x)\right| \leq \frac{1}{\Gamma\left(\nu\left(n-\mu\right)\right)}\int\limits_{a}^{x}\left(x-y\right)^{\nu(n-\mu)-1}\left|\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right|dy.$$

Using Hölder's inequality for $\{p,q\}$ the above inequality becomes

$$\begin{aligned}&\left|\left(D_{a+}^{\mu,\nu}f\right)(x)\right|\\ \leq\;& \frac{1}{\Gamma\left(\nu\left(n-\mu\right)\right)}\left(\int\limits_{a}^{x}\left(x-y\right)^{(\nu(n-\mu)-1)p}dy\right)^{1/p}\left(\int\limits_{a}^{x}\left|\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right|^{q}dy\right)^{1/q}\\ \leq\;& \frac{1}{\Gamma\left(\nu\left(n-\mu\right)\right)}\frac{\left(x-a\right)^{\nu(n-\mu)-1+\frac{1}{p}}}{\left(\left(\nu\left(n-\mu\right)-1\right)p+1\right)^{1/p}}\left(\int\limits_{a}^{b}\left|\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right|^{q}dy\right)^{1/q}.\end{aligned}$$

Thus we have

$$\left|\left(D_{a+}^{\mu,\nu}f\right)(x)\right|^{q} \leq \frac{1}{\left(\Gamma\left(\nu\left(n-\mu\right)\right)\right)^{q}}\frac{\left(x-a\right)^{q[\nu(n-\mu)-1]+\frac{q}{p}}}{\left(\left(\nu\left(n-\mu\right)-1\right)p+1\right)^{q/p}}\int\limits_{a}^{b}\left|\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(y)\right|^{q}dy.$$

Integrating both sides from a to b gives inequality (2.3).

If in particular we take $\nu=1$, we obtain the upcoming result given in [6].

Remark 2.4. Let $n-1<\mu<n$, $n\in\mathbb{N}$, $0<p<\frac{1}{1-(n-\mu)}$, $q>1$. If $f^{(n)}\in L^{q}\left(a,b\right)$, then the following inequality holds true:

$$\int\limits_{a}^{b}\left|\left({}^{C}D_{a+}^{\mu}f\right)(x)\right|^{q}dx \leq C\int\limits_{a}^{b}\left|f^{(n)}\left(y\right)\right|^{q}dy,$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $C=\frac{1}{(\Gamma(n-\mu))^{q}}\frac{(b-a)^{q(n-\mu)}}{(((n-\mu)-1)p+1)^{q/p}(q(n-\mu))}$.

The upcoming corollary is a special case of Theorem 1.2 for Hilfer fractional derivative.

Corollary 2.5. Let u be a weight function on (a,b) and let $n-1<\mu<n, 0<\nu\leq 1, n\in\mathbb{N}$. Let $f\in L\left(a,b\right)$ and define v on (a,b) by

$$v\left(y\right)=\nu\left(n-\mu\right)\int\limits_{y}^{b}u\left(x\right)\frac{\left(x-y\right)^{\nu(n-\mu)-1}}{\left(x-a\right)^{\nu(n-\mu)}}dx<\infty.$$

If $\varphi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\,\varphi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left|\left(D_{a+}^{\mu,\nu} f\right)(x)\right|\right)dx \leq \int_a^b v(y)\,\varphi\left(\left|\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right|\right)dy \tag{2.4}$$

holds true.

Proof. Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases}$$

and $K(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}$. Replace f by $D_{a+}^{\mu+\nu(n-\mu)} f$ and $g(x) = \left(D_{a+}^{\mu,\nu} f\right)(x)$, we obtain inequality (2.4).

Specially for $\nu = 1$, we obtain Corollary 2.9 of [6].

Remark 2.6. Choose $u(x) = (x-a)^{\nu(n-\mu)}$ a particular weight on (a, b) in Corollary 2.5, then we obtain the following inequality:

$$\int_a^b (x-a)^{\nu(n-\mu)}\varphi\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left|\left(D_{a_+}^{\mu,\nu} f\right)(x)\right|\right)dx \leq \int_a^b (b-y)^{\nu(n-\mu)}\varphi\left(\left|\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right|\right)dy. \tag{2.5}$$

Although (1.3) holds for all convex and increasing functions but some choices of φ are of particular interest. Namely, we shall consider power function. Let $q > 1$ and the function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\varphi(x) = x^q$, then (2.5) reduces to

$$\int_a^b (x-a)^{\nu(n-\mu)}\left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}}\left|\left(D_{a_+}^{\mu,\nu} f\right)(x)\right|\right)^q dx \leq \int_a^b (b-y)^{\nu(n-\mu)}\left|\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right|^q dy. \tag{2.6}$$

Since $x \in (a,b)$ and $\nu(n-\mu)(1-q) < 0$, then we obtain that the left-hand side of (2.6) satisfies

$$\int_a^b (x-a)^{\nu(n-\mu)} \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left| \left(D_{a+}^{\mu,\nu} f \right)(x) \right| \right)^q dx$$

$$\geq \quad (b-a)^{\nu(n-\mu)(1-q)} \left(\Gamma(\nu(n-\mu))+1 \right)^q \int_a^b \left| \left(D_{a+}^{\mu,\nu} f \right)(x) \right|^q dx, \tag{2.7}$$

and the right-hand side of (2.6) satisfies

$$\int_a^b (b-y)^{\nu(n-\mu)} \left(\left| \left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right| \right)^q dy$$

$$\leq (b-a)^{\nu(n-\mu)} \int_a^b \left(\left| \left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right| \right)^q dy. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8) we get

$$\int_a^b \left| \left(D_{a+}^{\mu,\nu} f \right)(x) \right|^q dx \leq \left(\frac{(b-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)} \right)^q \int_a^b \left(\left| \left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right| \right)^q dy.$$

Taking power $\frac{1}{q}$ on both sides, we obtain

$$\|D_{a+}^{\mu,\nu} f\|_q \leq \left(\frac{(b-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)} \right) \|D_{a+}^{\mu+\nu(n-\mu)} f\|_q.$$

Remark 2.7. In particular if $\nu = 0$ inequality (2.6) represents inequality of G. H. Hardy for Riemann-Liouvile fractional derivative of order μ and for $\nu = 1$ it becomes inequality of G. H. Hardy for Caputo fractional derivative of order μ.

Corollary 2.8. Let u be a weight function on (a,b) and let $n-1 < \mu < n,\ 0 < \nu \leq 1,\ n \in \mathbb{N}$. Define v on (a,b) as:

$$v(y) = \frac{(D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y)}{\Gamma(\nu(n-\mu))} \int_y^b u(x) \frac{(x-y)^{\nu(n-\mu)-1}}{(D_{a_+}^{\mu,\nu} f_2)(x)} dx < \infty.$$

If $\varphi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\varphi\left(\left| \frac{\left(D_{a_+}^{\mu,\nu} f_1 \right)(x)}{\left(D_{a_+}^{\mu,\nu} f_2 \right)(x)} \right| \right) dx \leq \int_a^b v(y)\varphi\left(\frac{\left| (D_{a_+}^{\mu+\nu(n-\mu)} f_1)(y) \right|}{\left| (D_{a_+}^{\mu+\nu(n-\mu)} f_2)(y) \right|} \right) dy \tag{2.9}$$

holds true for all $f_i \in L^1[a,b]$.

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \le y \le x \\ 0, & x \le y \le b, \end{cases}$$

and replace f_i by $D_{a+}^{\mu+\nu(n-\mu)} f_i$, $(i = 1,2)$ and $g_i = D_{a+}^{\mu,\nu} f_i$, $(i = 1,2)$, we get inequality (2.9).

In particular if we take $\nu = 1$, we obtain Corollary 3.14 of [13].
The upcoming corollary is the generalization of Corollary 2.5.

Corollary 2.9. Let $0 < p \le q < \infty, u$ be a weight function on $(a,b), n-1 < \mu < n$, $0 < \nu \le 1$, $n \in \mathbb{N}$. Let $D_{a+}^{\mu,\nu}$ be the left sided Hilfer fractional derivative and v is defined on (a,b) by

$$v(y) := \nu(n-\mu) \left[\int_y^b u(x) \left(\frac{(x-y)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}} \right)^{\frac{q}{p}} dx \right]^{\frac{p}{q}} < \infty.$$

If φ is a non-negative increasing convex function on an interval $I \subseteq \mathbb{R}$, then the following inequality

$$\left[\int_a^b u(x) \left(\varphi \left(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} \left(D_{a+}^{\mu,\nu} f \right)(x) \right) \right)^{\frac{q}{p}} dx \right]^{\frac{1}{q}} \le \left[\int_a^b v(y) \varphi \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f \right)(y) \right) dy \right]^{\frac{1}{p}} \tag{2.10}$$

holds true for all measurable functions $f : (a,b) \to \mathbb{R}$ such that $Im D_{a+}^{\mu+\nu(n-\mu)} f \subseteq I$.

Proof. Applying Theorem 1.4 with $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases}$$

and $K(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}$. Replace f by $D_{a+}^{\mu+\nu(n-\mu)} f$ and $g(x) = \left(D_{a+}^{\mu,\nu} f \right)(x)$, we obtain inequality (2.10).

Example 2.10. If $D_{a+}^{\mu,\nu}$ is the Hilfer fractional derivative, $u(x) = (x-y)^{\frac{\nu(n-\mu)q}{p}}$ is a particular weight and $\varphi(x) = x^s$, $s \ge 1, x > 0$ is convex, then after some calculation we obtain the following inequality:

$$\left[\int_a^b \left(D_{a+}^{\mu,\nu} f(x)\right)^{\frac{sq}{p}} dx\right]^{\frac{1}{q}}$$

$$\leq \frac{(\nu(n-\mu))^{\frac{1}{p}} (b-a)^{\frac{q(\nu(n-\mu)s-1)+p}{pq}}}{\left((\nu(n-\mu)\frac{q}{p}+1)\right)^{\frac{1}{q}} (\Gamma(\nu(n-\mu)+1))^{\frac{s}{p}}}$$

$$\times \left(\int_a^b \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f\right)(y)\right)^s dy\right)^{\frac{1}{p}}.$$

Theorem 2.11. Let u be a weight function, $n-1<\mu<n$, $0<\nu\leq 1$, $n\in\mathbb{N}$ and let $D_{a+}^{\mu,\nu} f$ be the Hilfer fractional derivative. If $f_i\in L[a,b], 0<a<b<\infty$ and $x\mapsto \frac{u(x)\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(y)(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))\left(D_{a+}^{\mu,\nu}f_2\right)(x)}$ is integrable over (a,b), then $v(y)$ is defined by

$$v(y)=\frac{\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(y)}{\Gamma(\nu(n-\mu))}\int_y^b u(x)\frac{(x-y)^{\nu(n-\mu)-1}}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}dx.$$

If $\varphi:(0,\infty)\times(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\varphi\left(\left|\frac{\left(D_{a+}^{\mu,\nu}f_1\right)(x)}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\right|,\left|\frac{\left(D_{a+}^{\mu,\nu}f_3\right)(x)}{\left(D_{a+}^{\mu,\nu}f_2\right)(x)}\right|\right)dx$$

$$\leq \int_a^b v(y)\varphi\left(\left|\frac{\left(D_{a+}^{\mu+\nu(n-\mu)}f_1\right)(y)}{\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(y)}\right|,\left|\frac{\left(D_{a+}^{\mu+\nu(n-\mu)}f_3\right)(y)}{\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(y)}\right|\right)dy \tag{2.11}$$

holds true.

Proof. Applying Theorem 1.5 with with $\Omega_1=\Omega_2=(a,b), d\mu_1(x)=dx, d\mu_2(y)=dy$

$$k(x,y)=\begin{cases}\frac{(x-y)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a\leq y\leq x;\\ 0, & x<y\leq b,\end{cases}$$

and $K(x)=\frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}$. Replace f_i by $D_{a+}^{\mu+\nu(n-\mu)}f_i$ and $g_i(x)=\left(D_{a+}^{\mu,\nu}f_i\right)(x)$ to obtain inequality (2.11).

Remark 2.12. In particular if we choose $\nu=1$ in Theorem 2.11, then $v(y)$ can be written as:

$$v(y)=\frac{f_2^{(n)}(y)}{\Gamma((n-\mu))}\int_y^b u(x)\frac{(x-y)^{(n-\mu)-1}}{\left(D_{*a}^{\mu}f_2\right)(x)}dx,$$

and inequality (2.11) represents results for Caputo fractional derivative which is given as:

$$\int_a^b u(x)\varphi\left(\left|\frac{\left(D^{\mu}_{*a}f_1\right)(x)}{\left(D^{\mu}_{*a}f_2\right)(x)}\right|,\left|\frac{\left(D^{\mu}_{*a}f_3\right)(x)}{\left(D^{\mu}_{*a}f_2\right)(x)}\right|\right)dx \leq \int_a^b v(y)\varphi\left(\left|\frac{f_1^{(n)}(y)}{f_2^{(n)}(y)}\right|,\left|\frac{f_3^{(n)}(y)}{f_2^{(n)}(y)}\right|\right)dy.$$

3 Hardy-type Inequalities for generalized fractional derivative involving Mittag-Leffler function in its kernel

First, we survey some facts about fractional integral operator which contain 6 parameter Mittag-Leffler function in the kernal (see [14, p. 1-13]), then we present Hardy-type inequalities for that integral operator.

Definition 3.1. Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}; min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)\}>0; p,q>0$ and $q<\Re\alpha+p$, then the integral operator defined by

$$\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}g\right)(x)=\int_a^x (x-y)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-y)^{\alpha})g(y)dy,$$

which contains the generalized Mittag-Leffler function

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_{pn}}, \tag{3.1}$$

in its kernel is investigated and its boundedness is proved under certain conditions. The function (3.1) represents all the previous generalizations of Mittag-Leffler function by setting

- $p=q=1$, it reduces to $E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum\limits_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{(\delta)_n}$ defined by Salim in [15].
- $\delta=p=1$, it represents $E^{\gamma,q}_{\alpha,\beta}(z)=\sum\limits_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$ which was introduced by A. K. Shukla and J. C. Prajapati in [16]. In [17] H. M. Srivastava and Z. Tomovski investigated the properties of this function and its existence for a wider set of parameters.
- $\delta=p=q=1$, the operator (3.1) is defined by Prabhakar in [18] and is denoted as: $E^{\gamma}_{\alpha,\beta}(z)=\sum\limits_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$.
- $\gamma=\delta=p=q=1$, it reduces to Wiman's function presented in [19], moreover if $\beta=1$, Mittag-Leffler function $E_{\alpha}(z)$ will be the result.

Lemma 3.2. Let $\alpha,\beta,\gamma,\delta,\omega>0; p,q>0, q<\Re\alpha+p$ and take $e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x)=x^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega x^{\alpha})$. Then the following integral holds true:

$$\int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x-y)\,dy=e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a).$$

Proof. Consider

$$\begin{aligned}\int_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x-y)\,dy &= \int_a^x (x-y)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-y)^\alpha)dy \\ &= \int_a^x (x-y)^{\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\left(w(x-y)^\alpha\right)^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}dy \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,(w)^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\int_a^x (x-y)^{\alpha n+\beta-1}dy \\ &= (x-a)^\beta\,E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha) \\ &= e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a)\,.\end{aligned}$$

This completes the lemma. It also holds for 3 parameter functions $e^{\gamma}_{\alpha,\beta,\omega}(x)=x^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega x^\alpha)$.

The upcoming corollary represents Theorem 1.2 for fractional integral involving Mittag-Leffler function in its kernel.

Corollary 3.3. Let u be a weight function on (a,b) and $\alpha,\beta,\gamma>0$. Let $f\in L(a,b)$ and v is defined on (a,b) by

$$v(y)=\int_y^b u(x)\,\frac{(x-y)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega(x-y)^\alpha)}{(x-a)^\beta E^{\gamma}_{\alpha,\beta+1}(\omega(x-a)^\alpha)}dx<\infty.$$

If $\varphi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the following inequality

$$\int_a^b u(x)\,\varphi\left(\frac{\left|\left(\varepsilon^{\gamma}_{\alpha,\beta,\omega,a+}f\right)(x)\right|}{(x-a)^\beta E^{\gamma}_{\alpha,\beta+1}(\omega(x-a)^\alpha)}\right)dx\le\int_a^b v(y)\,\varphi(|f(y)|)\,dy \tag{3.2}$$

holds.

Proof. Applying Theorem 1.2 with $\Omega_1=\Omega_2=(a,b), d\mu_1(x)=dx, d\mu_2(y)=dy$

$$k(x,y)=\begin{cases}(x-y)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega(x-y)^\alpha), & a\le y\le x;\\ 0, & x<y\le b,\end{cases}$$

$K(x)=(x-a)^\beta E^{\gamma}_{\alpha,\beta+1}(\omega(x-a)^\alpha)$. Taking $g(x)=\left(\varepsilon^{\gamma}_{\alpha,\beta,\omega,a+}f\right)(x)$, we get inequality (3.2).

Next result is an extension of Corollary 3.3.

Corollary 3.4. Let u be a weight function on (a,b) and let $\alpha,\beta,\gamma,\delta,\omega$ be positive real numbers. Also $p,q>0$ and $q<\Re\alpha+p$. Let $f\in L(a,b)$ and define v on (a,b) by

$$v(y)=\int_y^b u(x)\,\frac{(x-y)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-y)^\alpha)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}dx<\infty.$$

If $\varphi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\,\varphi\left(\frac{\left|\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f\right)(x)\right|}{(x-a)^{\beta} E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\left(\omega (x-a)^{\alpha}\right)}\right) dx \le \int_a^b v(y)\,\varphi\left(|f(y)|\right) dy \tag{3.3}$$

holds true.

Proof. Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a,b)\,, d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} (x-y)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\left(\omega (x-y)^{\alpha}\right), & a \le y \le x; \\ 0, & x \le y \le b, \end{cases}$$

and $K(x) = (x-a)^{\beta} E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\left(\omega (x-a)^{\alpha}\right)$. Taking $g(x) = \left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f\right)(x)$, we get inequality (3.3).

Corollary 3.5. Let the assumptions of Corollary 3.4 be satisfied and v on (a,b) is defined by

$$v(y) = f_2(y) \int_y^b u(x) \frac{(x-y)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\left(w(x-y)^{\alpha}\right)}{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2(x)} dx.$$

If $\varphi : (0, \infty) \to \mathbb{R}$ is convex and increasing function, then the following inequality

$$\int_a^b u(x)\varphi\left(\left|\frac{\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1\right)(x)}{\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2\right)(x)}\right|\right) dx \le \int_a^b v(y)\varphi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy \tag{3.4}$$

holds for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $(i = 1, 2)$.

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} (x-y)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\left(w(x-y)^{\alpha}\right), & a \le y \le x \\ 0, & x \le y \le b, \end{cases}$$

$K(x) = (x-a)^{\beta} E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\left(\omega (x-a)^{\alpha}\right)$. Taking $g_i(x) = \left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_i\right)(x)$, we get inequality (3.4).

Remark 3.6. If we choose $\omega = 0$ in Corollary 3.5, then we obtained Corollary 3.11 of [7, p. 44] for left sided Riemann-Liouville fractional integral operator.

We next give Hardy-type inequality for generalized fractional integral operator involving generalized Mittag-Leffler function in its kernel.

Theorem 3.7. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha, \beta, \gamma, \delta, \omega > 0$. If $f \in L^q(a,b)\,, 0 < a < b < \infty$, then the inequality

$$\int_a^b \left|\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a+} f\right)(x)\right|^q dx \le C \int_a^b |f(x)|^q\, dx$$

holds true, where $C = \left[e^{\gamma,\delta,q}_{\alpha,\beta+2,p,\omega}(b-a)\right]^q$.

Proof. Applying Hölder's inequality, we have

$$
\begin{aligned}
\left|\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a+}f\right)(x)\right| &\leq \int\limits_a^x \left|e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x-y)\right| |f(y)|\, dy \\
&\leq \left(\int\limits_a^x \left|e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x-y)\right|^p dy\right)^{1/p} \left(\int\limits_a^x |f(y)|^q\, dy\right)^{1/q} \\
&\leq \left(\int\limits_a^x e^{\gamma,\delta,q}_{\alpha,\beta,p,\omega}(x-y)\, dy\right) \left(\int\limits_a^b |f(x)|^q\, dx\right)^{1/q} \\
&= \left[e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a)\right] \left(\int\limits_a^b |f(x)|^q\, dx\right)^{1/q}.
\end{aligned}
$$

Thus we have

$$
\left|\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a+}f\right)(x)\right|^q \leq \left[e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a)\right]^q \left(\int\limits_a^b |f(x)|^q\, dx\right),
$$

for every $x \in [a,b]$. Integrating on both sides from a to b, we get

$$
\begin{aligned}
\int\limits_a^b \left|\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a+}f(x)\right|^q dx &\leq \left(\int\limits_a^b \left[e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a)\right]^q dx\right) \left(\int\limits_a^b |f(x)|^q\, dx\right) \\
&\leq \left(\int\limits_a^b e^{\gamma,\delta,q}_{\alpha,\beta+1,p,\omega}(x-a)\, dx\right)^q \left(\int\limits_a^b |f(x)|^q\, dx\right).
\end{aligned}
$$

Applying Lemma 3.2, we obtain

$$
\int\limits_a^b \left|\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a+}f(x)\right|^q dx \leq \left(e^{\gamma,\delta,q}_{\alpha,\beta+2,p,\omega}(b-a)\right)^q \left(\int\limits_a^b |f(x)|^q\, dx\right).
$$

Corollary 3.8. Let $u(x)$ be a weight function on (a,b) and let $\alpha,\beta,\gamma,\delta,\omega > 0; p,q > 0, q < \Re\alpha + p$. Define v on (a,b) by

$$
v(y) = \left[\int\limits_y^b u(x) \left(\frac{(x-y)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-y)^\alpha)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)} dx\right)^{\frac{q}{p}}\right]^{\frac{p}{q}} < \infty.
$$

If $\varphi : I \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\left[\int_a^b u(x)\left(\varphi\left(\frac{\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f\right)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\left(\omega\left(x-a\right)^{\alpha}\right)}\right)\right)^{\frac{q}{p}}dx\right]^{\frac{1}{q}}$$

$$\leq \left(\int_a^b v\left(y\right)\varphi\left(\left|f\left(y\right)\right|\right)dy\right)^{\frac{1}{p}} \tag{3.5}$$

holds true for all measurable functions $f : (a.b) \to \mathbb{R}$ such that $Imf \subseteq I$.

Proof. Applying Theorem 1.4 with $\Omega_1 = \Omega_2 = (a,b), d\mu_1(x) = dx, d\mu_2(y) = dy$

$$k(x,y) = \begin{cases} (x-y)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}\left(w(x-y)^{\alpha}\right), & a \leq y \leq x \\ 0, & x \leq y \leq b, \end{cases}$$

$g = \varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f$, we get inequality (3.5).

The upcoming result is an application of Theorem 1.5.

Corollary 3.9. Let u be a weight function and $\alpha, \beta, \gamma, \delta, \omega$ be the positive real numbers. If $f_i \in L\left(a,b\right), 0 < a < b < \infty, p, q > 0$, then $v(y)$ is defined by

$$v(y) = f_2(y)\int_y^b \frac{u(x)(x-y)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-y)^{\alpha})}{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2(x)}dx.$$

If $\varphi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\varphi\left(\left|\frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_1(x)}{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2(x)}\right|,\left|\frac{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_3(x)}{\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f_2(x)}\right|\right)dx \leq \int_a^b \varphi\left(\left|\frac{f_1(y)}{f_2(y)}\right|,\left|\frac{f_3(y)}{f_2(y)}\right|\right)dy \tag{3.6}$$

holds true.

Remark 3.10. If in particular we choose $\omega = 0$ in Corollary 3.9, then $v(y)$ becomes

$$v(y) = \frac{f_2(y)}{\Gamma(\beta)}\int_y^b \frac{u(x)(x-y)^{\beta-1}}{I^{\beta}_{a^+}f_2(x)}dx,$$

and inequality (3.6) can be written as:

$$\int_a^b u(x)\varphi\left(\left|\frac{I^{\beta}_{a^+}f_1(x)}{I^{\beta}_{a^+}f_2(x)}\right|,\left|\frac{I^{\beta}_{a^+}f_3(x)}{I^{\beta}_{a^+}f_2(x)}\right|\right)dx \leq \int_a^b \varphi\left(\left|\frac{f_1(y)}{f_2(y)}\right|,\left|\frac{f_3(y)}{f_2(y)}\right|\right)dy.$$

Acknowledgements. The research of J. Pečarić has been fully supported by Croatian Science Foundation under the project 5435. The author Zivorad Tomovski was supported under the European Commission and the Croatian Ministry of Science, Education and Sports Co-Financing agreement No. 291823. In particular Z. Tomovski acknowledge project financing from the Maria Curie FP-7-PEOPLE-2011-COFUND program NEWFELPRO Grant Agreement No. 37 (Anomalous diffusion).

References

[1] Hardy GH. Notes on some points in the integral calculus, LX. An inequality between integrals. Messenger of Math. 1925;54:150-156.

[2] Abramovich S, Krulic K, Pečarić J, Persson LE. Some new refined Hardy type inequalities with general kernels and measures. Aequationes Math. 2010;79:157-172.

[3] Čižmešija A, Krulić K, Pečarić J. Some new refined Hardy-type inequalities with kernels. J Math Inequal. 2010;4(4):481-503.

[4] Čižmešija A, Krulić K, Pečarić J. A new class of general refined Hardy-type inequality with kernels. Rad HAZU. 2013;17:53-80.

[5] Farid G, Krulic K, Pecaric J. On refinement of Hardy type inequalities via superquadratic functions. Sarajevo J Math. 2011;7(20):163-175.

[6] Iqbal S, Krulić K, Pečarić J. On an inequality of H. G. Hardy. J Inequal Appl. Volume 2010:Article ID 264347.

[7] Krulić K, Pečarić J, Persson LE. Some new Hardy type inequalities with general kernels. Math Inequal Appl. 2009;12:473-485.

[8] Niculescu C, Persson LE. Convex functions and their applications. A contemporary approach. CMC Books in Mathematics: Springer, New York; 2006.

[9] Iqbal S, Krulić K, Pečarić J. On an inequality for convex function with some applications on fractional derivatives and fractional integrals. J Math Inequal. 2011;5(2):219-230.

[10] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204: Elsevier. New York-London; 2006.

[11] Tomovski Ž, Hilfer R, Srivastava HM. Fractional and Operational Calculus with Generalized Fractional Derivative Operators and Mittag-Leffler Type Functions. Integral Transforms Spec Funct. 2010;21(11):797-814.

[12] Hilfer R, Luchko Y, Tomovski Ž. Operational method for solution of fractional differential equations with generalized Riemann-Liouville fractional derivative. Fractional Calculus & Applied Analysis. 2009;12(3):299-318.

[13] Krulić K, Pečarić J, Pokaz D. Inequalities of Hardy and Jensen; 2013.

[14] Salim TO, Faraj AW. A generalization of Mittag-Leffler function and integral operator associated with fractional calculus. JFCA. 2012;5:1-13.

[15] Salim TO. Some properties relating to the generalized Mittag-Leffler function. Adv Appl Math Anal. 2009;4:21-30.

[16] Shukla AK, Prajapati JC. On a generalization of Mittag-Leffler function and its properties. J Math Anal Appl. 2007;336:797-811.

[17] Srivastava HM, Tomovski Z. Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernal. Appl Math Comput. 2009;211:198-210.

[18] Prabhakar TR. A Singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math J. 1971;19:7-15.

[19] Wiman A. Über den fundamentalsatz in der theorie der Funktionen. Acta Math. 1905;29:191-201.